

# A SUM FORMULA OF MULTIPLE $L$ -VALUES

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ABSTRACT. We show a sum formula of certain multiple  $L$ -values conjectured by Essouabri-Matsumoto-Tsumura, which generalizes the sum formula of multiple zeta-values. The proof relies on the method of partial fraction decomposition.

## 1. MAIN RESULTS

Let  $\mathbb{N} = \{1, 2, \dots\}$  be the set of positive integers. We take  $n, k \in \mathbb{N}$  such that  $k \geq n + 1$  and put

$$I(k, r) = \left\{ \mathbf{k} = (k_1, \dots, k_n) \in \mathbb{N}^n \mid \begin{array}{l} k_n \geq 2, \ k_1 + \dots + k_n = k, \\ k_1 = \dots = k_{r-1} = 1 \end{array} \right\}$$

for  $r = 1, 2, \dots, n, n + 1$ . In particular,  $I(k, 1)$  is the set of admissible indices for multiple zeta-values

$$\zeta(\mathbf{k}) = \sum_{m_1, \dots, m_n=1}^{\infty} \frac{1}{m_1^{k_1} (m_1 + m_2)^{k_2} \dots (m_1 + \dots + m_n)^{k_n}}$$

of weight  $k$  and depth  $n$ , whereas  $I(k, n + 1) = \emptyset$ .

We also consider the multiple  $L$ -values of the following type:

$$L(\mathbf{k}, f, J) = \sum_{m_1, \dots, m_n=1}^{\infty} \frac{f(\sum_{j \in J} m_j)}{m_1^{k_1} (m_1 + m_2)^{k_2} \dots (m_1 + \dots + m_n)^{k_n}}.$$

Here  $f: \mathbb{N} \rightarrow \mathbb{C}$  is a function and  $J$  is a nonempty subset of  $\{1, \dots, n\}$ . Observe that, when  $f$  is periodic and  $J$  is of the form  $\{j\}$  or  $\{1, \dots, j\}$ , then these  $L(\mathbf{k}, f, J)$  may be viewed as special cases of the multiple  $L$ -values introduced by Arakawa-Kaneko [1].

Our main result is the following:

**Theorem 1.1.** *The sum formula*

$$(1.1) \quad \sum_{\emptyset \neq J \subset \{1, \dots, n\}} \sum_{\mathbf{k} \in I(k, \max J)} (-1)^{|J|-1} L(\mathbf{k}, f, J) = \sum_{m=1}^{\infty} \frac{f(m)}{m^k}$$

holds. Here  $f: \mathbb{N} \rightarrow \mathbb{C}$  is a function such that  $L(\mathbf{k}, f, J)$  are absolutely convergent for all  $J$  and all  $\mathbf{k} \in I(k, \max J)$ .

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A sufficient condition for absolute convergence is that  $f(m) = O(m^{k-n-\varepsilon})$  for some  $\varepsilon > 0$ . In fact, if we put  $r = \max J$ , then any  $\mathbf{k} = (k_1, \dots, k_n) \in I(k, r)$  satisfies  $(k_r - 1) + \dots + (k_n - 1) = k - n$ . Hence we have

$$\begin{aligned}
& \sum_{m_1, \dots, m_n=1}^{\infty} \frac{|f(\sum_{j \in J} m_j)|}{m_1^{k_1} \dots (m_1 + \dots + m_n)^{k_n}} \\
& \leq C \sum_{m_1, \dots, m_n=1}^{\infty} \frac{(\sum_{j \in J} m_j)^{k-n-\varepsilon}}{m_1^{k_1} \dots (m_1 + \dots + m_n)^{k_n}} \\
& \leq C \sum_{m_1, \dots, m_n=1}^{\infty} \frac{(m_1 + \dots + m_r)^{k_r-1} \dots (m_1 + \dots + m_n)^{k_n-1-\varepsilon}}{m_1^{k_1} \dots (m_1 + \dots + m_n)^{k_n}} \\
& = C \sum_{m_1, \dots, m_n=1}^{\infty} \frac{1}{m_1 \dots (m_1 + \dots + m_{n-1})(m_1 + \dots + m_n)^{1+\varepsilon}}
\end{aligned}$$

for some constant  $C$ , and it is well-known that the rightmost side is convergent.

The formula (1.1) was shown by Essouabri-Matsumoto-Tsumura [2] for  $n = 2$  and 3 and conjectured for general  $n$ . As they remarked, when  $f(m) = 1$  identically, (1.1) reduces to the sum formula for multiple zeta-values

$$(1.2) \quad \sum_{\mathbf{k} \in I(k, 1)} \zeta(\mathbf{k}) = \zeta(k).$$

Our proof of (1.1), which relies essentially on the partial fraction decomposition such as  $\frac{1}{ab} = \frac{1}{a+b}(\frac{1}{a} + \frac{1}{b})$ , is similar to that of (1.2) by Granville [3].

To prove (1.1), it suffices to consider functions  $f$  supported at a single element  $m \in \mathbb{N}$ . To state explicitly, we put

$$\begin{aligned}
M(m, J) &= \left\{ \mathbf{m} = (m_1, \dots, m_n) \in \mathbb{N} \mid \sum_{j \in J} m_j = m \right\}, \\
S(k, m, J) &= \sum_{\mathbf{k} \in I(k, \max J)} \sum_{\mathbf{m} \in M(m, J)} \frac{1}{m_1^{k_1} (m_1 + m_2)^{k_2} \dots (m_1 + \dots + m_n)^{k_n}}
\end{aligned}$$

for  $m \in \mathbb{N}$  and  $\emptyset \neq J \subset \{1, \dots, n\}$ . Then Theorem 1.1 is equivalent to the following:

**Theorem 1.2.** *For any  $m \in \mathbb{N}$ , we have*

$$(1.3) \quad \sum_{\emptyset \neq J \subset \{1, \dots, n\}} (-1)^{|J|-1} S(k, m, J) = \frac{1}{m^k}.$$

There is another equivalent form of our main theorem. In fact, the generating function argument implies that Theorem 1.2 is equivalent to the equation (1.1) for the function  $f(m) = t^m$ , where  $t$  ranges over all complex numbers with  $|t| \leq 1$ . In this case, the left hand side of (1.1) is

$$(1.4) \quad \sum_{\emptyset \neq J \subset \{1, \dots, n\}} \sum_{\mathbf{k} \in I(k, \max J)} \sum_{m_1, \dots, m_n=1}^{\infty} \frac{(-1)^{|J|-1} t^{\sum_{j \in J} m_j}}{m_1^{k_1} \dots (m_1 + \dots + m_n)^{k_n}}.$$

Now let us put  $I'(k, r) = I(k, r) \setminus I(k, r+1)$  ( $r = 1, \dots, n$ ). If we fix  $\mathbf{k} \in I'(k, r)$  and  $m_1, \dots, m_n \geq 1$ , the corresponding sum of numerators in (1.4) is

$$\begin{aligned} \sum_{J \neq \emptyset, \max J \leq r} (-1)^{|J|-1} t^{\sum_{j \in J} m_j} &= - \sum_{J \neq \emptyset, \max J \leq r} \prod_{j \in J} (-t^{m_j}) \\ &= -\{(1 - t^{m_1}) \cdots (1 - t^{m_r}) - 1\} \\ &= 1 - (1 - t^{m_1}) \cdots (1 - t^{m_r}). \end{aligned}$$

Therefore, the equation (1.1) becomes

$$(1.5) \quad \sum_{r=1}^n \sum_{\mathbf{k} \in I'(k, r)} \sum_{m_1, \dots, m_n=1}^{\infty} \frac{1 - (1 - t^{m_1}) \cdots (1 - t^{m_r})}{m_1^{k_1} \cdots (m_1 + \cdots + m_n)^{k_n}} = \sum_{m=1}^{\infty} \frac{t^m}{m^k}.$$

We can also obtain the following more symmetric form, by taking the difference of (1.5) and itself for  $t = 1$  (the latter is just the sum formula (1.2)).

**Theorem 1.3.** *For any complex number  $t$  such that  $|t| \leq 1$ , we have*

$$(1.6) \quad \sum_{r=1}^n \sum_{\mathbf{k} \in I'(k, r)} \sum_{m_1, \dots, m_n=1}^{\infty} \frac{(1 - t^{m_1}) \cdots (1 - t^{m_r})}{m_1^{k_1} \cdots (m_1 + \cdots + m_n)^{k_n}} = \sum_{m=1}^{\infty} \frac{1 - t^m}{m^k}.$$

From the next section, we will show (1.3) by computing the left hand side in three steps:

- (i) For  $1 \leq l \leq r \leq n$ , compute the sum over  $J$  such that  $|J| = l$  and  $\max J = r$  (Proposition 2.1).
- (ii) Sum up the values of (i) for  $r = l, \dots, n$  (Proposition 3.2).
- (iii) Sum up alternately the values of (ii) for  $l = 1, \dots, n$ .

## 2. FIRST STEP

Fix  $n, k \in \mathbb{N}$  such that  $k \geq n+1$ , and  $m \in \mathbb{N}$ . For  $l, r \in \mathbb{N}$  such that  $1 \leq l \leq r \leq n$ , we put

$$\begin{aligned} A_l &= \sum_{\substack{(p_1, \dots, p_{l-1}) \in \mathbb{N}^{l-1} \\ p_1 + \cdots + p_{l-1} < m}} \frac{1}{p_1(p_1 + p_2) \cdots (p_1 + \cdots + p_{l-1})}, \\ B_{l,r}(p_l, \dots, p_{r-1}) &= \frac{1}{p_l(p_l + p_{l+1}) \cdots (p_l + \cdots + p_{r-1})}, \end{aligned}$$

and

$$\begin{aligned} C_{l,r}(p_l, \dots, p_{n-1}, k_r, \dots, k_n) \\ = \frac{1}{(m + p_l + \cdots + p_{r-1})^{k_r} \cdots (m + p_l + \cdots + p_{n-1})^{k_n}}. \end{aligned}$$

Note that

$$A_l = \sum_{\substack{(p_1, \dots, p_{l-1}) \in \mathbb{N}^{l-1} \\ p_1 + \cdots + p_{l-1} < m}} B_{1,l}(p_1, \dots, p_{l-1}).$$

**Proposition 2.1.**

$$\sum_{\substack{|J|=l \\ \max J=r}} S(k, m, J) = A_l \sum_{\mathbf{p}=(p_l, \dots, p_{n-1}) \in \mathbb{N}^{n-l}} B_{l,r}(\mathbf{p}) \sum_{\mathbf{k} \in I(k, r)} C_{l,r}(\mathbf{p}, \mathbf{k}).$$

**Remark 2.2.** Here the notation  $B_{l,r}(\mathbf{p})$  for  $\mathbf{p} = (p_l, \dots, p_{n-1}) \in \mathbb{N}^{n-l}$  means  $B_{l,r}(p_l, \dots, p_{r-1})$ . In other words, to evaluate  $B_{l,r}$  at  $\mathbf{p}$ , we simply omit the redundant components  $p_r, \dots, p_{n-1}$ . We also use similar notations in the following.

*Proof.* We consider shuffles of  $\{1, \dots, l-1\}$  and  $\{l, \dots, r-1\}$ , i.e., bijections  $\sigma$  of the set  $\{1, \dots, r-1\}$  onto itself which are monotone increasing on subsets  $\{1, \dots, l-1\}$  and  $\{l, \dots, r-1\}$ . If we denote the set of all such  $\sigma$  by  $\text{Sh}(l, r)$ , we have a map

$$\begin{aligned} \text{Sh}(l, r) &\longrightarrow \{J \subset \{1, \dots, n\} \mid |J| = l, \max J = r\} \\ \sigma &\longmapsto \sigma(\{1, \dots, l-1\}) \cup \{r\} \end{aligned}$$

which is clearly bijective.

If  $\sigma \in \text{Sh}(l, r)$  corresponds to  $J$ , each  $\mathbf{m} = (m_1, \dots, m_n) \in M(m, J)$  can be expressed as

$$m_j = \begin{cases} p_{\sigma^{-1}(j)} & (1 \leq j \leq r-1), \\ m - (p_1 + \dots + p_{l-1}) & (j = r), \\ p_{j-1} & (j > r) \end{cases}$$

by a unique  $\mathbf{p} = (p_1, \dots, p_{n-1}) \in \mathbb{N}^{n-1}$  such that  $p_1 + \dots + p_{l-1} < m$ . If this is the case, we have

$$m_1 + m_2 + \dots + m_j = m + p_l + \dots + p_{j-1}$$

for  $j \geq r$ . Hence we can rewrite the definition of  $S(k, m, J)$  as

$$\begin{aligned} &S(k, m, J) \\ &= \sum_{\mathbf{k} \in I(k, r)} \sum_{\substack{\mathbf{p} \in \mathbb{N}^{n-1} \\ p_1 + \dots + p_{l-1} < m}} \frac{1}{p_{\sigma^{-1}(1)} \cdots (p_{\sigma^{-1}(1)} + \dots + p_{\sigma^{-1}(r-1)})} \\ (2.1) \quad &\quad \times \frac{1}{(m + p_l + \dots + p_{r-1})^{k_r} \cdots (m + p_l + \dots + p_{n-1})^{k_n}} \\ &= \sum_{\substack{\mathbf{p} \in \mathbb{N}^{n-1} \\ p_1 + \dots + p_{l-1} < m}} \frac{1}{p_{\sigma^{-1}(1)} \cdots (p_{\sigma^{-1}(1)} + \dots + p_{\sigma^{-1}(r-1)})} \sum_{\mathbf{k} \in I(k, r)} C_{l,r}(\mathbf{p}, \mathbf{k}). \end{aligned}$$

To sum up (2.1) over all  $\sigma \in \text{Sh}(l, r)$ , we need the following lemma:

**Lemma 2.3.** *Let  $s$  and  $t$  be non-negative integers, and let  $x_1, \dots, x_s, y_1, \dots, y_t$  be  $s+t$  indeterminates. When  $(z_1, \dots, z_{s+t})$  runs over all shuffles of  $(x_1, \dots, x_s)$  and  $(y_1, \dots, y_t)$  (i.e., permutations of  $\{x_1, \dots, x_s, y_1, \dots, y_t\}$  which preserve the order of  $\{x_1, \dots, x_s\}$  and  $\{y_1, \dots, y_t\}$  respectively), the summation formula*

$$\begin{aligned} (2.2) \quad &\sum_{(z_1, \dots, z_{s+t})} \frac{1}{z_1(z_1 + z_2) \cdots (z_1 + \dots + z_{s+t})} \\ &= \frac{1}{x_1(x_1 + x_2) \cdots (x_1 + \dots + x_s)} \frac{1}{y_1(y_1 + y_2) \cdots (y_1 + \dots + y_t)} \end{aligned}$$

holds.

*Proof.* We use the induction on the pair  $(s, t)$ . If  $s$  or  $t$  is zero, the formula is trivial. On the other hand, when  $s, t \geq 1$ , all shuffles  $(z_1, \dots, z_{s+t})$  are divided into two classes, namely, the ones such that  $z_{s+t} = x_s$  and the ones such that  $z_{s+t} = y_t$ . The former shuffles are in one-to-one correspondence with shuffles of  $(x_1, \dots, x_{s-1})$  and  $(y_1, \dots, y_t)$  in an obvious manner, and similarly for the latter. Hence, by the

induction hypothesis for  $(s-1, t)$  and  $(s, t-1)$ , the left hand side of (2.2) is equal to

$$\begin{aligned} & \left( \frac{1}{x_1 \cdots (x_1 + \cdots + x_{s-1})} \frac{1}{y_1 \cdots (y_1 + \cdots + y_t)} \right. \\ & \quad \left. + \frac{1}{x_1 \cdots (x_1 + \cdots + x_s)} \frac{1}{y_1 \cdots (y_1 + \cdots + y_{t-1})} \right) \\ & \quad \times \frac{1}{x_1 + \cdots + x_s + y_1 + \cdots + y_t}. \end{aligned}$$

Then the equality

$$\begin{aligned} & \left( \frac{1}{y_1 + \cdots + y_t} + \frac{1}{x_1 + \cdots + x_s} \right) \frac{1}{x_1 + \cdots + x_s + y_1 + \cdots + y_t} \\ & \quad = \frac{1}{(x_1 + \cdots + x_s)(y_1 + \cdots + y_t)}, \end{aligned}$$

shows that the formula (2.2) holds for  $(s, t)$ .  $\square$

Let us return to the proof of Proposition 2.1. By Lemma 2.3, the equality

$$\begin{aligned} & \sum_{\sigma \in \text{Sh}(l, r)} \frac{1}{p_{\sigma^{-1}(1)} \cdots (p_{\sigma^{-1}(1)} + \cdots + p_{\sigma^{-1}(r-1)})} \\ & \quad = \frac{1}{p_1(p_1 + p_2) \cdots (p_1 + \cdots + p_{l-1})} \frac{1}{p_l(p_l + p_{l+1}) \cdots (p_l + \cdots + p_{r-1})} \\ & \quad = B_{1,l}(\mathbf{p}) B_{l,r}(\mathbf{p}) \end{aligned}$$

holds for each  $\mathbf{p} = (p_1, \dots, p_{n-1}) \in \mathbb{N}^{n-1}$ . Therefore, by summing up (2.1), we conclude

$$\begin{aligned} \sum_{\substack{|J|=l \\ \max J=r}} S(k, m, J) &= \sum_{\substack{\mathbf{p}=(p_1, \dots, p_{n-1}) \in \mathbb{N}^{n-1} \\ p_1 + \cdots + p_{l-1} < m}} B_{1,l}(\mathbf{p}) B_{l,r}(\mathbf{p}) \sum_{\mathbf{k} \in I(k, r)} C_{l,r}(\mathbf{p}, \mathbf{k}) \\ &= A_l \sum_{\mathbf{p}=(p_l, \dots, p_{n-1}) \in \mathbb{N}^{n-l}} B_{l,r}(\mathbf{p}) \sum_{\mathbf{k} \in I(k, r)} C_{l,r}(\mathbf{p}, \mathbf{k}) \end{aligned}$$

as required.  $\square$

### 3. SECOND STEP

The purpose in this section is to compute  $\sum_{|J|=l} S(k, m, J)$  for  $1 \leq l \leq n$ . By Proposition 2.1, we have

$$(3.1) \quad \sum_{|J|=l} S(k, m, J) = A_l \sum_{\mathbf{p} \in \mathbb{N}^{n-l}} D_l(\mathbf{p}),$$

where we put

$$D_l(\mathbf{p}) = \sum_{r=l}^n B_{l,r}(\mathbf{p}) \sum_{\mathbf{k} \in I(k, r)} C_{l,r}(\mathbf{p}, \mathbf{k}).$$

**Lemma 3.1.** For  $t = l, \dots, n-1$ ,

$$(3.2) \quad \sum_{r=l}^t B_{l,r}(\mathbf{p}) \sum_{\mathbf{k} \in I(k, r)} C_{l,r}(\mathbf{p}, \mathbf{k}) = B_{l,t}(\mathbf{p}) \sum_{\mathbf{k} \in I(k, t)} \frac{1}{m^{k_t}} C_{l,t+1}(\mathbf{p}, \mathbf{k}).$$

*Proof.* We use the induction on  $t$ . When  $t = l$ , the claim is obvious since

$$C_{l,l}(\mathbf{p}, \mathbf{k}) = \frac{1}{m^{k_l}} C_{l,l+1}(\mathbf{p}, \mathbf{k}).$$

Let  $t \geq l + 1$ . Then any element of  $I(k, t - 1)$  can be uniquely expressed as

$$(k_1, \dots, k_{t-2}, i, k_t + 1 - i, k_{t+1}, \dots, k_n)$$

by  $(k_1, \dots, k_n) \in I(k, t)$  and  $1 \leq i \leq k_t$ . Therefore, by the induction hypothesis for  $t - 1$ , we have

$$\begin{aligned} & \sum_{r=l}^t B_{l,r}(\mathbf{p}) \sum_{\mathbf{k} \in I(k,r)} C_{l,r}(\mathbf{p}, \mathbf{k}) \\ &= B_{l,t-1}(\mathbf{p}) \sum_{\mathbf{k} \in I(k,t-1)} \frac{1}{m^{k_{t-1}}} C_{l,t}(\mathbf{p}, \mathbf{k}) + B_{l,t}(\mathbf{p}) \sum_{\mathbf{k} \in I(k,t)} C_{l,t}(\mathbf{p}, \mathbf{k}) \\ &= B_{l,t}(\mathbf{p}) \sum_{\mathbf{k} \in I(k,t)} \left( \sum_{i=1}^{k_t} \frac{1}{m^i} \frac{p_l + \dots + p_{t-1}}{(m + p_l + \dots + p_{t-1})^{k_t+1-i}} \right. \\ & \quad \left. + \frac{1}{(m + p_l + \dots + p_{t-1})^{k_t}} \right) C_{l,t+1}(\mathbf{p}, \mathbf{k}). \end{aligned}$$

Now, from the equality

$$\sum_{i=1}^K \frac{1}{x^i} \frac{y}{(x+y)^{K+1-i}} + \frac{1}{(x+y)^K} = \frac{1}{x^K},$$

our claim (3.2) follows immediately.  $\square$

By Lemma 3.1 for  $t = n - 1$ ,  $D_l(\mathbf{p})$  can be written as

$$D_l(\mathbf{p}) = B_{l,n-1}(\mathbf{p}) \sum_{\mathbf{k} \in I(k,n-1)} \frac{1}{m^{k_{n-1}}} C_{l,n}(\mathbf{p}, \mathbf{k}) + B_{l,n}(\mathbf{p}) \sum_{\mathbf{k} \in I(k,n)} C_{l,n}(\mathbf{p}, \mathbf{k}).$$

By definition, we see that

$$C_{l,n}(\mathbf{p}, \mathbf{k}) = \frac{1}{(m + p_l + \dots + p_{n-1})^{k_n}}$$

and

$$\begin{aligned} I(k, n - 1) &= \{(1, \dots, 1, i, k - (n - 2) - i) \mid 1 \leq i \leq k - n\}, \\ I(k, n) &= \{(1, \dots, 1, 1, k - (n - 1))\}. \end{aligned}$$

Hence a computation similar to the proof of Lemma 3.1 shows that

$$\begin{aligned}
 D_l(\mathbf{p}) &= B_{l,n}(\mathbf{p}) \left( \sum_{i=1}^{k-n} \frac{1}{m^i} \frac{p_l + \cdots + p_{n-1}}{(m + p_l + \cdots + p_{n-1})^{k-(n-2)-i}} \right. \\
 &\quad \left. + \frac{1}{(m + p_l + \cdots + p_{n-1})^{k-(n-1)}} \right) \\
 &= B_{l,n}(\mathbf{p}) \frac{1}{m^{k-n}} \frac{1}{m + p_l + \cdots + p_{n-1}} \\
 &= B_{l,n-1}(\mathbf{p}) \frac{1}{m^{k-n}} \frac{1}{(p_l + \cdots + p_{n-1})(m + p_l + \cdots + p_{n-1})} \\
 (3.3) \quad &= B_{l,n-1}(\mathbf{p}) \frac{1}{m^{k-n+1}} \left( \frac{1}{p_l + \cdots + p_{n-1}} - \frac{1}{m + p_l + \cdots + p_{n-1}} \right).
 \end{aligned}$$

**Proposition 3.2.** For  $l = 1, \dots, n$ ,

$$\sum_{|J|=l} S(k, m, J) = \frac{1}{m^{k-(n-1)}} \sum_{\substack{m > q_1 > \cdots > q_{l-1} \geq 1 \\ 1 \leq q_l \leq \cdots \leq q_{n-1} \leq m}} \frac{1}{q_1 q_2 \cdots q_{n-1}}.$$

*Proof.* First, it is obvious from the definition that

$$A_l = \sum_{m > q_1 > \cdots > q_{l-1} \geq 1} \frac{1}{q_1 q_2 \cdots q_{l-1}}.$$

Hence, by (3.1) and (3.3), it suffices to show

$$\begin{aligned}
 (3.4) \quad \sum_{\mathbf{p} \in \mathbb{N}^{n-l}} B_{l,n-1}(\mathbf{p}) \left( \frac{1}{p_l + \cdots + p_{n-1}} - \frac{1}{m + p_l + \cdots + p_{n-1}} \right) \\
 = \sum_{1 \leq q_l \leq \cdots \leq q_{n-1} \leq m} \frac{1}{q_l q_{l+1} \cdots q_{n-1}}.
 \end{aligned}$$

This is done by computing the summation for  $p_{n-1}, p_{n-2}, \dots, p_l$  successively. In fact, the first summation is

$$\begin{aligned}
 &\sum_{p_{n-1}=1}^{\infty} B_{l,n-1}(\mathbf{p}) \left( \frac{1}{p_l + \cdots + p_{n-1}} - \frac{1}{m + p_l + \cdots + p_{n-1}} \right) \\
 &= \sum_{p_{n-1}=1}^m B_{l,n-1}(\mathbf{p}) \frac{1}{p_l + \cdots + p_{n-1}} \\
 &= \sum_{p_{n-1}=1}^m B_{l,n-2}(\mathbf{p}) \frac{1}{p_{n-1}} \left( \frac{1}{p_l + \cdots + p_{n-2}} - \frac{1}{p_l + \cdots + p_{n-1}} \right).
 \end{aligned}$$

Then the second summation is

$$\begin{aligned}
& \sum_{p_{n-2}=1}^{\infty} \sum_{p_{n-1}=1}^m B_{l,n-2}(\mathbf{p}) \frac{1}{p_{n-1}} \left( \frac{1}{p_l + \cdots + p_{n-2}} - \frac{1}{p_l + \cdots + p_{n-1}} \right) \\
&= \sum_{p_{n-1}=1}^m \sum_{p_{n-2}=1}^{p_{n-1}} B_{l,n-2}(\mathbf{p}) \frac{1}{p_{n-1}} \frac{1}{p_l + \cdots + p_{n-2}} \\
&= \sum_{p_{n-1}=1}^m \sum_{p_{n-2}=1}^{p_{n-1}} B_{l,n-3}(\mathbf{p}) \frac{1}{p_{n-2}p_{n-1}} \left( \frac{1}{p_l + \cdots + p_{n-3}} - \frac{1}{p_l + \cdots + p_{n-2}} \right)
\end{aligned}$$

and so on. The last summation is

$$\begin{aligned}
& \sum_{p_l=1}^{\infty} \sum_{1 \leq p_{l+1} \leq \cdots \leq p_{n-1} \leq m} B_{l,l}(\mathbf{p}) \frac{1}{p_{l+1} \cdots p_{n-1}} \left( \frac{1}{p_l} - \frac{1}{p_l + p_{l+1}} \right) \\
&= \sum_{1 \leq p_l \leq \cdots \leq p_{n-1} \leq m} \frac{1}{p_l p_{l+1} \cdots p_{n-1}},
\end{aligned}$$

which is the right hand side of (3.4). This proves Proposition 3.2.  $\square$

#### 4. THIRD STEP

For  $l = 1, \dots, n$ , put

$$Q_l = \left\{ \mathbf{q} = (q_1, \dots, q_{n-1}) \in \mathbb{N}^{n-1} \mid \begin{array}{l} m > q_1 > \cdots > q_{l-1} \geq 1, \\ 1 \leq q_l \leq \cdots \leq q_{n-1} \leq m \end{array} \right\}.$$

Then Proposition 3.2 says that

$$(4.1) \quad \sum_{|J|=l} S(k, m, J) = \frac{1}{m^{k-(n-1)}} \sum_{\mathbf{q} \in Q_l} \frac{1}{q_1 q_2 \cdots q_{n-1}}.$$

On the other hand, it is easy to see that

$$Q_1 = \{(m, \dots, m)\} \amalg (Q_1 \cap Q_2)$$

and

$$Q_l = (Q_l \cap Q_{l-1}) \amalg (Q_l \cap Q_{l+1}) \quad (2 \leq l \leq n),$$

where we put  $Q_{n+1} = \emptyset$ . Hence the inclusion-exclusion argument implies

$$(4.2) \quad \frac{1}{m^{n-1}} = \sum_{l=1}^n (-1)^{l-1} \sum_{\mathbf{q} \in Q_l} \frac{1}{q_1 q_2 \cdots q_{n-1}}.$$

Combining (4.1) and (4.2), we obtain

$$\begin{aligned}
\sum_{\emptyset \neq J \subset \{1, \dots, n\}} (-1)^{|J|-1} S(k, m, J) &= \sum_{l=1}^n (-1)^{l-1} \sum_{|J|=l} S(k, m, J) \\
&= \frac{1}{m^{k-(n-1)}} \sum_{l=1}^n (-1)^{l-1} \sum_{\mathbf{q} \in Q_l} \frac{1}{q_1 q_2 \cdots q_{n-1}} \\
&= \frac{1}{m^k}.
\end{aligned}$$

Now the proof of Theorem 1.2 is complete.



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